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AN ANALYTIC, ONE-PARAMETER FAMILY OF SELF-ADJOINT BOUNDARY CONDITIONS FOR SCHRÖDINGER OPERATORS ON AN INTERVAL

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ABSTRACT

A one-parameter family of real, homogeneous boundary conditions on the interval [0,1], under which the operator $-\frac{d^2}{dx^2}$ is self-adjoint, is constructed. The relation between such boundary conditions and Lagrangian planes in $\mathbb{R}^4$ is used and the resulting circle of boundary conditions is seen to include Dirichlet, Neumann, periodic, antiperiodic, and several other well-known examples.

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SIGNIFICANCE AND EXPLANATION

A more or less explicit deformation of boundary conditions for an interval is constructed, under which the operator $-\frac{d^2}{dx^2}$ remains self-adjoint. The deformation depends analytically on its parameter and includes Dirichlet, Neumann, periodic, antiperiodic, and other boundary conditions. It is hoped that this family of self-adjoint boundary conditions can be used to construct solutions in problems where one set of boundary conditions (for example, periodic or Dirichlet) leads to a significant simplification of the problem.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

AN ANALYTIC, ONE-PARAMETER FAMILY OF SELF-ADJOINT BOUNDARY CONDITIONS FOR SCHRÖDINGER OPERATORS ON AN INTERVAL

Robert L. Sachs

Consider the operator $L \equiv -\dfrac{d^2}{dx^2}$ acting on reasonably nice functions which satisfy the pair of real linear homogenous boundary conditions

(1) $\quad a_i y(0) + b_i y'(0) + c_i y(1) + d_i y'(1) = 0, \qquad i = 1,2.$

By definition, this operator is self-adjoint if and only if the bilinear form:

(2) $\quad B(y,z) \equiv y(0)\, z'(0) - y'(0)z(0) - y(1)z'(1) + y'(1)z(1)$

vanishes identically for all u,v satisfying the boundary conditions (1). In terms of column vectors $Y \equiv (y(0),y'(1),y'(0),y(1))^T$, $Z \equiv (z(0),z'(1),z'(0),z(1))^T$, (2) is equivalent to

(3) $\quad Y^T J Z = 0$ where J is the usual 4×4 symplectic matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ where $0,I$ are 2×2 matrices. If Y,Z are in the span of

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \\ \delta_1 \end{pmatrix}, \quad \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \\ \delta_2 \end{pmatrix}$$

then (3) is equivalent to $(A^T\ B^T)J\binom{A}{B} = 0$ where

$A \equiv \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}$, $B \equiv \begin{pmatrix} \gamma_1 & \gamma_2 \\ \delta_1 & \delta_2 \end{pmatrix}$ and this leads immediately to the condition

(4) $$A^T B - B^T A = 0$$

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which is clearly no stronger than the requirement

(5) $$A + iB \text{ is unitary.}$$

We shall construct a one-parameter family of unitary matrices $U(t)$ connecting the matrix representing periodic boundary conditions:

(6) $$y(0) = y(1); \quad y'(0) = y'(1)$$

with the matrix representing Dirichlet boundary conditions:

(7) $$y(0) = y(1) = 0.$$

(6) is equivalent to $Y \in \operatorname{span}\left\{ \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix} \right\}$ which leads to the unitary matrix $\begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ while Dirichlet boundary conditions (7) are equivalent to the unitary matrix $\begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}$. Thus we seek a one-parameter family $U(t)$ of unitary matrices with

(8) $$U(0) = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}; \quad U(1) = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}.$$

Such a family is easily found in the form

(9) $$U(t) = U(0)e^{iSt} \quad \text{where} \quad e^{iS} = U(0)^{-1}U(1) = \begin{pmatrix} -i\sqrt{2} & i/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

i.e.

(10) $$iS = \log \begin{pmatrix} -i/\sqrt{2} & i/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{aligned} &= \log(U(0)^{-1}U(1)). \\ &= \log M \end{aligned}$$

Now $M = U(0)^{-1}\, U(1)$ has eigenvalues $\lambda = e^{-\pi i/4}(e^{\pm\pi i/3}) = e^{\pi i/12},\ e^{-7\pi i/12}$ and, diagonalizing M, we find

(11) $$M = P \begin{pmatrix} e^{\pi i/12} & 0 \\ 0 & e^{-7\pi i/12} \end{pmatrix} P^{-1}$$ where P, P^{-1} are given by the 2×2 matrices

(12) $$P = \begin{pmatrix} \frac{\sqrt{3}-1}{\sqrt{2}}\, e^{\pi i/4} & -\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right) e^{\pi i/4} \\ 1 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} \frac{e^{-\pi i/4}}{\sqrt{6}} & \frac{\sqrt{3}+1}{2\sqrt{3}} \\ \frac{-e^{-\pi i/4}}{\sqrt{6}} & \frac{\sqrt{3}-1}{2\sqrt{3}} \end{pmatrix}$$

Thus $$e^{iSt} = P \begin{pmatrix} e^{\pi i/12t} & 0 \\ 0 & e^{-7\pi i/12t} \end{pmatrix} P^{-1}$$

and the desired path is

(13) $$U(t) = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} e^{iSt}$$

which, by a tedious computation, is precisely the matrix

(14) $U(t) =$

$$\begin{pmatrix} \frac{e^{-\pi i/4t}}{\sqrt{2}} \left(\cos(\pi/3t) - \frac{\sin\pi/3t}{\sqrt{3}}\right) & \frac{e^{-\pi i/4t}}{\sqrt{2}} \left(i \cos \frac{\pi}{3}t + \left(\frac{-2+i}{\sqrt{3}}\right)\sin \frac{\pi}{3}\right) \\ \frac{e^{-\pi i/4t}}{\sqrt{2}} \left(i \cos(\pi/3t) + \frac{(2+i)}{\sqrt{3}}\sin\left(\frac{\pi}{t}t\right)\right) & \frac{e^{-\pi i/4t}}{\sqrt{2}}\left(\cos \pi/3t - \frac{1}{\sqrt{3}} \sin\pi/3t\right) \end{pmatrix}.$$

We see easily that U has period 24 and that $U(t + 12) = -U(t)$, indeed $U(t+6) = iU(t)$ so that, in terms of the corresponding boundary conditions, we need only consider $U(t)$, $0 \le t < 12$. Listing $U(j)$, $j = 0,\cdots,11$ and the corresponding boundary conditions, we have

$$
(15)\quad \left\{
\begin{array}{ll}
U(0) = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}; & U(1) = \begin{pmatrix} 0 & +i \\ 1 & 0 \end{pmatrix} \\[2ex]
U(2) = \begin{pmatrix} i/\sqrt{2} & i/\sqrt{2} \\ -i/\sqrt{2} & i/\sqrt{2} \end{pmatrix}; & U(3) = \begin{pmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1+i}{2} \end{pmatrix} \\[2ex]
U(4) = \begin{pmatrix} 0 & \frac{-1+i}{\sqrt{2}} \\ \frac{1+i}{\sqrt{2}} & 0 \end{pmatrix}; & U(5) = \begin{pmatrix} \frac{-1+i}{2} & \frac{-1+i}{2} \\ \frac{1-i}{2} & \frac{-1+i}{2} \end{pmatrix} \\[2ex]
U(6) = \begin{pmatrix} i/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & i/\sqrt{2} \end{pmatrix}; & U(7) = \begin{pmatrix} 0 & -1 \\ i & 0 \end{pmatrix} \\[2ex]
U(8) = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}; & U(9) = \begin{pmatrix} \frac{-1+i}{2} & \frac{-1-i}{2} \\ \frac{-1-i}{2} & \frac{-1+i}{2} \end{pmatrix} \\[2ex]
U(10) = \begin{pmatrix} 0 & \frac{-1-i}{\sqrt{2}} \\ \frac{-1+i}{\sqrt{2}} & 0 \end{pmatrix}; & U(11) = \begin{pmatrix} \frac{-1-i}{2} & \frac{-1-i}{2} \\ \frac{1+i}{2} & \frac{-1-i}{2} \end{pmatrix}
\end{array}
\right.
$$

The corresponding boundary conditions, found by reversing the procedure above, are as follows:

t	Boundary conditions
0	$y(0) = y(1)$, $y'(0) = y'(1)$
1	$y(0) = 0$, $y(1) = 0$
2	$y(0) = 0$, $y'(1) = 0$
3	$y(0) = -y'(1)$, $y'(0) = y(1)$
4	$y(0) = -y'(0)$, $y'(1) = y(1)$
5	$y(0) = -y'(0)$, $y'(1) = -y(1)$
6	$y(0) = -y(1)$, $y'(0) = -y'(1)$
7	$y'(0) = 0$, $y'(1) = 0$
8	$y'(0) = 0$, $y(1) = 0$
9	$y(0) = y'(1)$, $y'(0) = -y(1)$
10	$y(0) = y'(0)$, $y'(1) = -y(1)$
11	$y(0) = y'(0)$, $y'(1) = y(1)$

Thus our one parameter family in fact includes periodic, Dirichlet, antiperiodic, Neumann, and several other well-known boundary conditions.

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